Comparison of Implicit Discontinuous Galerkin and WENO Schemes on Stratigraphic and Unstructured Grids

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Introduction

- Field scale simulations: numerical diffusion masks important physics
  - Viscous fingering[1], shocks, misscible displacement[2]
  - Particularly evident in simulation of EOR[3] (polymer, solvent gas etc) and compositional behavior
- Counteracted by higher-order spatial discretizations
  - Continuous[4] and discontinuous[5] Galerkin methods, WENO[6], etc.
- Higher-order discretizations only used to limited extent on real reservoir models
  - Cumbersome to implement in implicit setting
  - Hard to formulate on irregular cell geometries
- Herein: discuss how WENO and Discontinuous Galerkin methods can be adapted to implicit reservoir simulation on unstructured and stratigraphic grids

[1] Riaz and Tchelepi, 2004 [2] Ewing et al., 1984 [3] Holing et al., 1990 [4] Arbogast and Wheeler, 1996
[5] Rivière and Wheeler, 2002 [6] Liu et. al, 1994

Why do we need implicit methods?

- Consider $\phi u_t + \vec{v} \cdot \nabla u = 0, \quad \vec{v} = -\frac{1}{\mu}\mathbf{K}\nabla p$
  - Large variations in petrophysical properties (porosity ϕ, permeability $\mathbf{K}$)
  - Large variations in $|\vec{v}|$: stagnant regions/high flow near wells

  $\rightarrow$ Explicit method: severe time-step restrictions, but significant smearing even with $CFL < 1$

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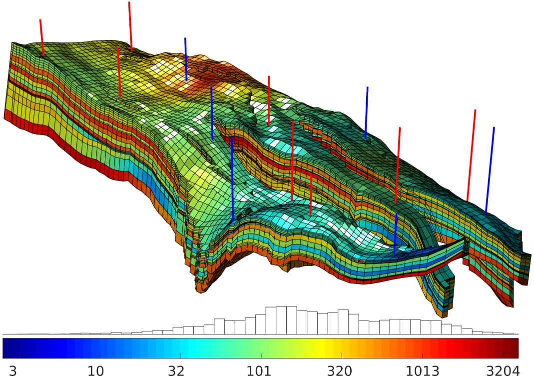
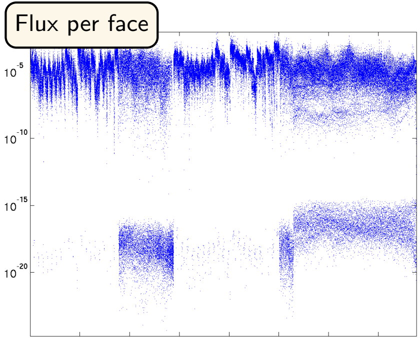




Discretization

- Discrete conservation of mass for phase α in cell i:

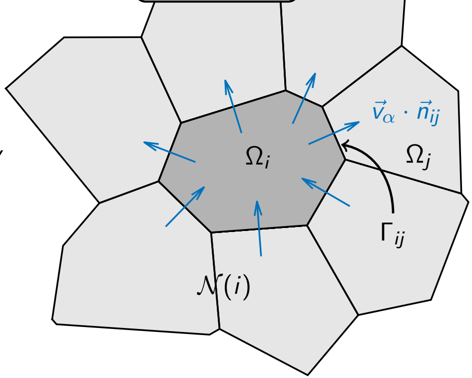
$$\frac{1}{\Delta t}\left[\mathcal{M}^i_\alpha(\mathbf{u}^{n+1},\psi) - \mathcal{M}^i_\alpha(\mathbf{u}^{n},\psi)\right] + \mathcal{F}^i_\alpha(\mathbf{u}^{n+1},\psi) - \mathcal{Q}^i_\alpha(\mathbf{u}^{n+1},\psi) = 0$$

Accumulation Flux Sources/sinks

$$\mathcal{M}^i_\alpha(\mathbf{u},\psi) = \int_{\Omega_i} [\phi \rho_\alpha S_\alpha]\psi \,\mathrm{d}V$$

$$\mathcal{F}^i_\alpha(\mathbf{u},\psi) = \sum_{j\in\mathcal{N}(i)} \int_{\Gamma_{ij}} [\rho_\alpha \vec{v}_\alpha \cdot \vec{n}_{ij}]\psi \,\mathrm{d}\sigma - \int_{\Omega_i} [\rho_\alpha \vec{v}_\alpha]\cdot\nabla\psi \,\mathrm{d}V$$

$$\mathcal{Q}^i_\alpha(\mathbf{u},\psi) = \int_{\Omega_i} q_\alpha \psi \,\mathrm{d}V$$



Discretization

- Flux terms needs special attention:

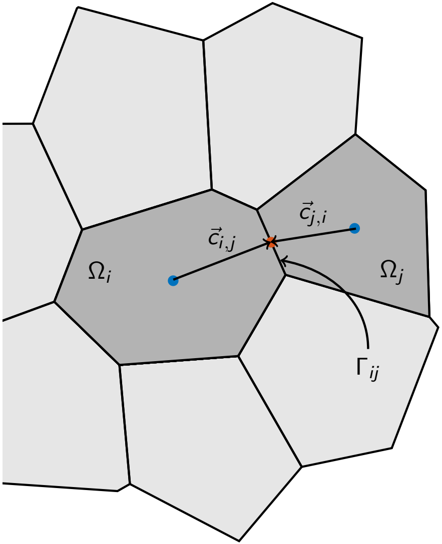
$$\mathcal{F}_\alpha^i(\mathbf{u},\psi) = \sum_{j\in\mathcal{N}(i)} \int_{\Gamma_{ij}} [\rho_\alpha \vec{v}_\alpha \cdot \vec{n}_{ij}]\psi \,\mathrm{d}\sigma - \int_{\Omega_i} [\rho_\alpha \vec{v}_\alpha] \cdot \nabla\psi \,\mathrm{d}V$$

- Interface fluxes given by Darcy's law

$$(\rho_\alpha \vec{v}_\alpha \cdot \vec{n})_{ij} = -(\rho_\alpha \lambda_\alpha \mathbf{K}\nabla p_\alpha \cdot \vec{n})_{ij}$$

- Two-point flux approximation:

$$(\mathbf{K}\nabla p_\alpha \cdot \vec{n})_{ij} = T_{ij}(p_j - p_i), \quad T_{ij} = \left(\frac{n_{i,j}^T \mathbf{K}_i \vec{c}_{i,j}}{|c_{i,j}|^2} + \frac{n_{j,i}^T \mathbf{K}_j \vec{c}_{j,i}}{|c_{j,i}|^2}\right)$$



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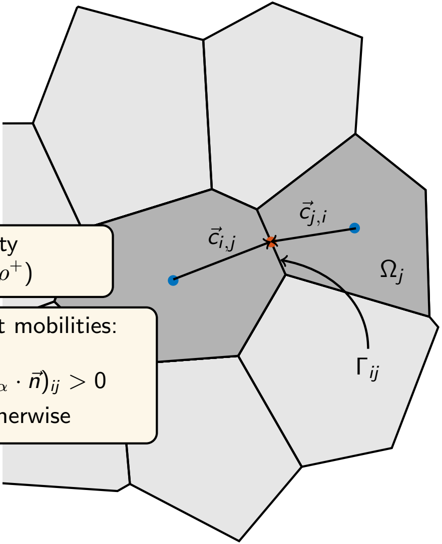
Averaged density
$$\rho_{ij} = \frac{1}{2}(\rho^- + \rho^+)$$

Upstream-weight mobilities:

$$\lambda_{ij} = \begin{cases} \lambda^- & (\vec{v}_\alpha \cdot \vec{n})_{ij} > 0 \\ \lambda^+ & \text{otherwise} \end{cases}$$

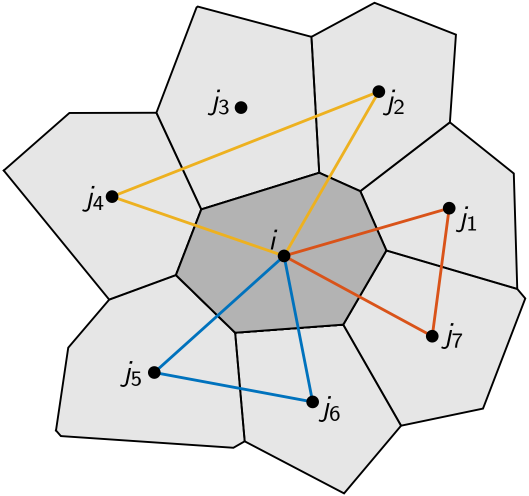


Weighted essentially non-oscillatory discretization (WENO)

- Second-order scheme: local interpolation of λ using neighbors $\mathcal{N}(i)$
- N neighbors $\rightarrow \binom{N}{2}$ planes (typically choose N planes)
- N gradient approximations

$$\{\sigma_i^k\}_{k=1}^N, \quad \sigma_i^k \approx \nabla \lambda_i$$

Primary stencils	Secondary stencils
(i, j_1, j_2)	(i, j_1, j_3)
(i, j_2, j_3)	(i, j_1, j_4)
(i, j_3, j_4)	(i, j_1, j_5)
(i, j_4, j_5)	(i, j_1, j_6)
(i, j_5, j_6)	(i, j_2, j_4)
(i, j_6, j_7)	(i, j_2, j_5)
(i, j_7, j_1)	(i, j_2, j_6)
	$\vdots$



Weighted essentially non-oscillatory discretization (WENO)

- Use planes to construct N linear reconstructions $\hat{\lambda}_i^k(\mathbf{x}) = \lambda_i + \sigma_i^k(\mathbf{x} - \mathbf{x}_i)$
- WENO: write reconstruction as convex combination

$$\hat{\lambda}_i(\mathbf{x}) = \sum_{k=1}^{N} w_i^k \hat{\lambda}_i^k(\mathbf{x})$$

- Given linear weights, $\sum_k \gamma_i^k = 1$, we compute **nonlinear weights**:

$$w_i^k = \beta_i^k / \sum_{k=1}^{N_i} \beta_i^k, \qquad \beta_i^k = \gamma_i^k / \left(\varepsilon + \Lambda_i^k\right)^2, \qquad \Lambda_i^k = |\vec{\sigma}_i^k|^2 \, |\Omega_i|$$

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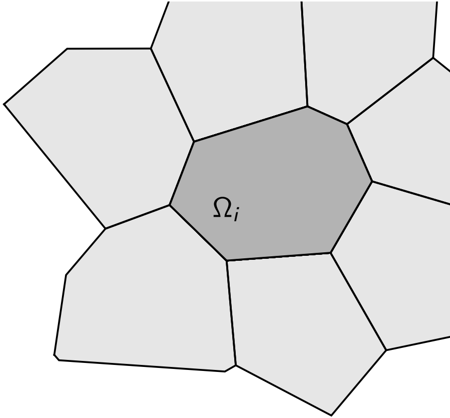
Smoothness indicator

Discontinuous Galerkin discretization (dG)

- dG(k): Choose basis functions $\{\psi\}$ basis for $\mathbb{P}_k$:

$$\frac{1}{\Delta t}\left[\mathcal{M}^i_\alpha(\mathbf{u}^{n+1},\psi) - \mathcal{M}^i_\alpha(\mathbf{u}^{n},\psi)\right] + \mathcal{F}^i_\alpha(\mathbf{u}^{n+1},\psi) - \mathcal{Q}^i_\alpha(\mathbf{u}^{n+1},\psi) = 0$$

- Unstructured and skewed cell geometries
  $\rightarrow$ Impractical to construct orthonormal basis functions



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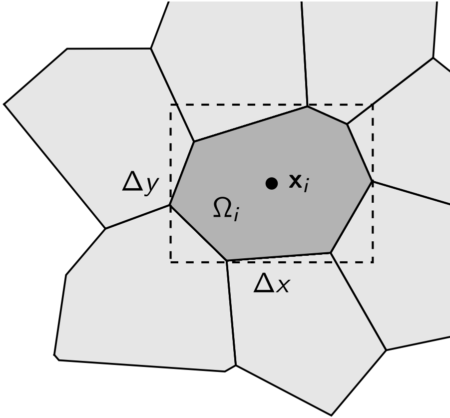
- Pragmatic solution: tensor products of Legendre polynomials on bounding box

$$\psi^i_j(\mathbf{x}) = \begin{cases} \ell_r\left(\frac{x-x_i}{\Delta x_i/2}\right)\ell_s\left(\frac{y-y_i}{\Delta y_i/2}\right)\ell_t\left(\frac{z-z_i}{\Delta z_i/2}\right), & \mathbf{x}\in\Omega_i, \\ 0, & \mathbf{x}\notin\Omega_i, \end{cases}$$



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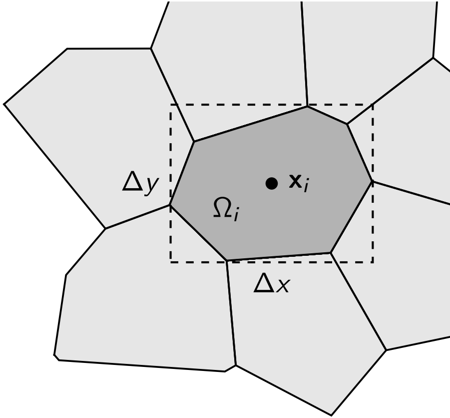
- Simple "limiter" strategy: reduce to dG(0)
  - if values outside physical range
  - if jump across cell interfaces $> \varepsilon$



Discontinuous Galerkin discretization (dG)

- How to evaluate integrals over irregular cell geometries?
  $\rightarrow$ Cubature rules by *moment fitting*

$$\begin{bmatrix} \psi_1^i(\xi_1) & \dots & \psi_1^i(\xi_m) \\ \vdots & \ddots & \vdots \\ \psi_n^i(\xi_1) & \dots & \psi_n^i(\xi_m) \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix} = \frac{1}{|\Omega|} \underbrace{\begin{bmatrix} \int_\Omega \psi_1^i \, \mathrm{d}V \\ \vdots \\ \int_\Omega \psi_n^i \, \mathrm{d}V \end{bmatrix}}$$

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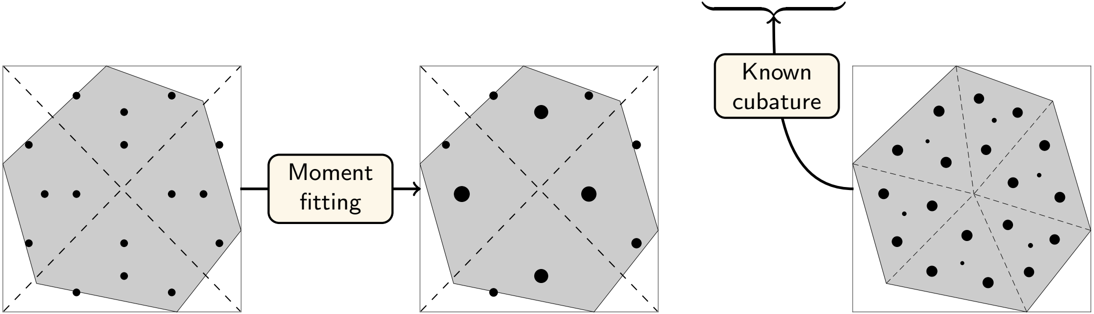


Autmatic differentiation

- Linearize and neglect higher-order terms $\rightarrow$ Newton-Raphson method:

$$\mathbf{u}^{k+1} = \mathbf{u}^{k} + \Delta\mathbf{u}, \quad -\mathbf{J}\Delta\mathbf{u} = \mathcal{R}$$

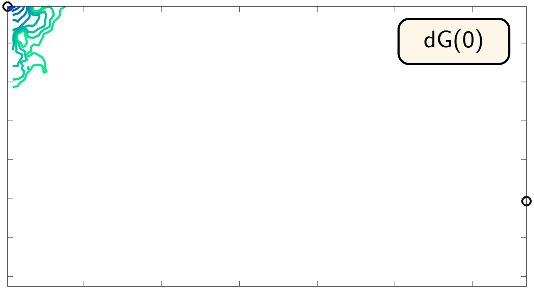
Jacobian
$\mathbf{J}_{i,j} = \partial\mathcal{R}_i/\partial\mathbf{u}_j$

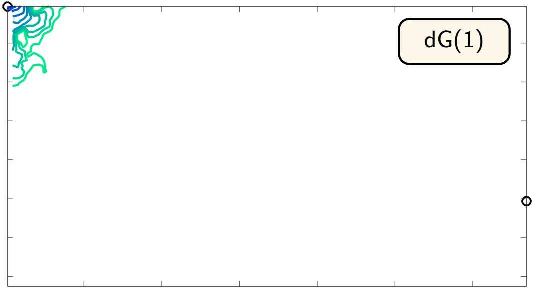
Residual

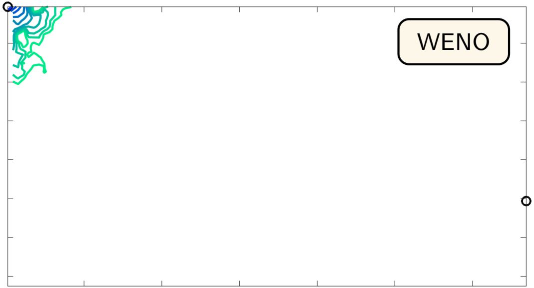
- Calculation of Jacobian $\mathbf{J}$: involved and error-prone process $\rightarrow$ **automatic differentiation**
  - Facilitates implementation of higher-order schemes and nonlinear interpolations

Automatic differentiation: evaluation of residuals consist of a nested sequence of elementary binary operations and unary operations. Expand each variable with data elements representing derivatives wrt. all primary variables. Combine chain rule and elementary differentiation rules to analytically evaluate all derivatives. All you have to do is code the residual evaluation, and then Jacobians are computed simultaneously by operator overloading.

Example 2: subset of SPE 10 Model 2







- Layer 70 of SPE10 Model 2
- Fluvial sandstone channels on mudstone background
- Quadratic relperms, slightly compressible fluids
- Injection of water over 2000 days

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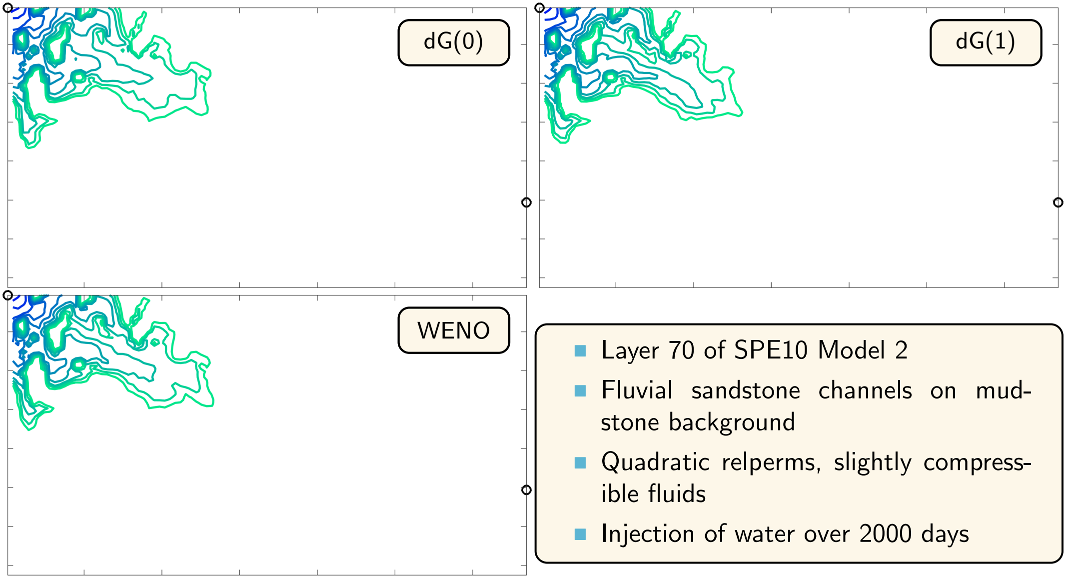

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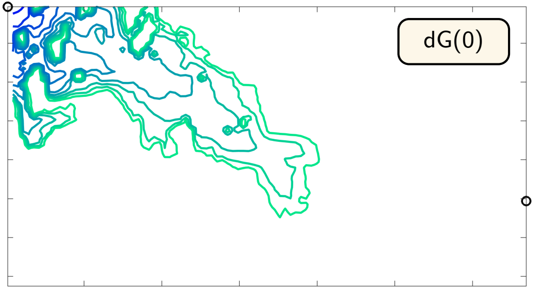

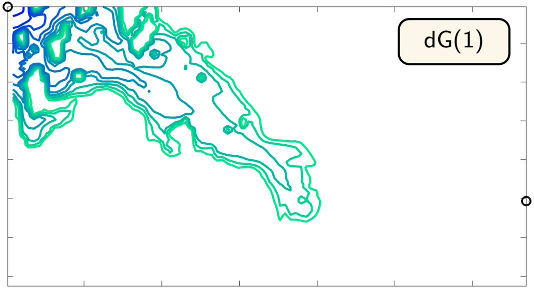

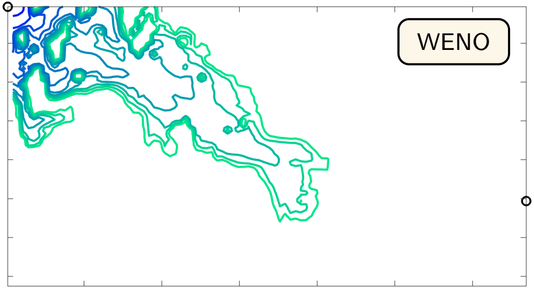

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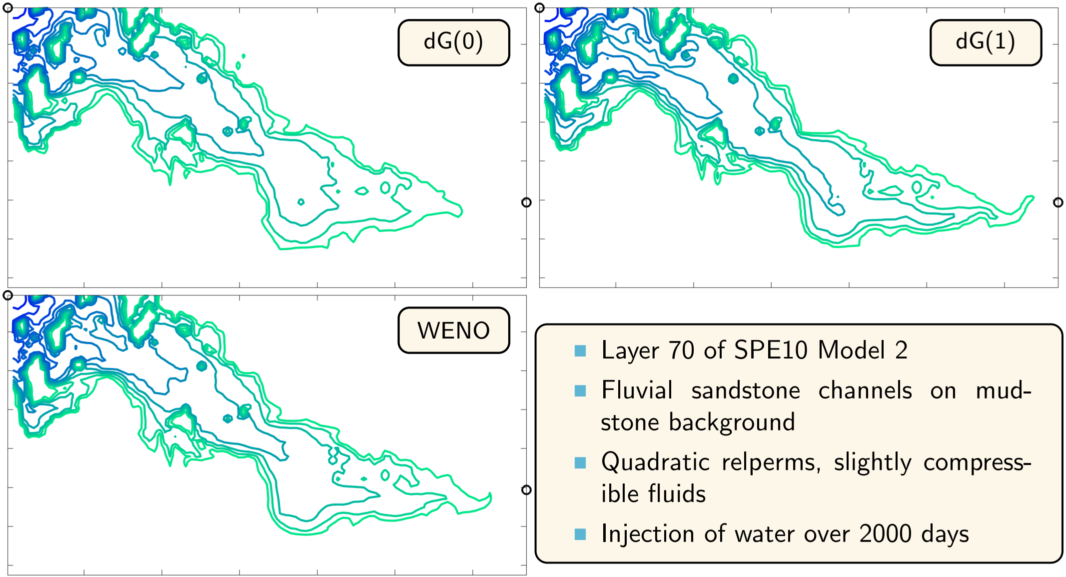

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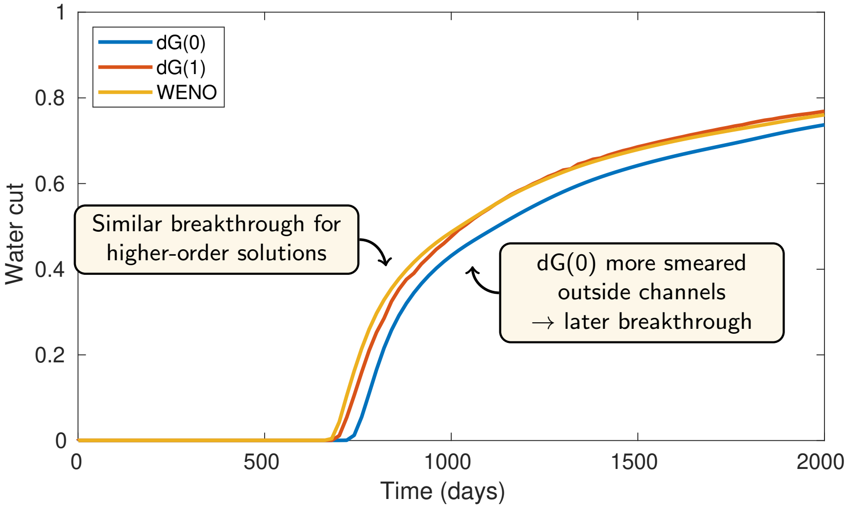
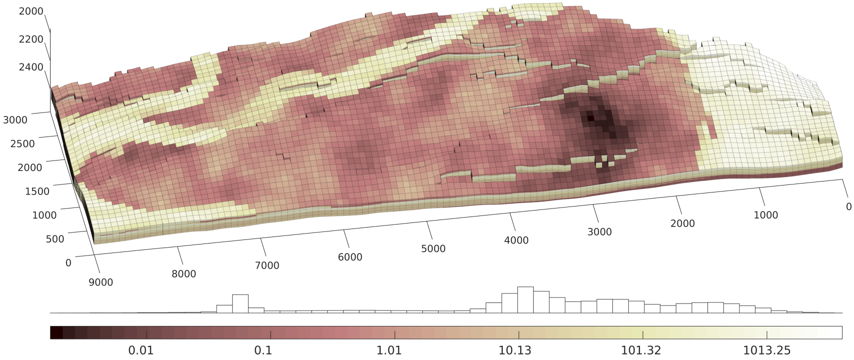


Subset of SAIGUP realization



- Shallow-marine oil reservoir, modeled in the SAIGUP study[1]
- Spans lateral area of $\sim 9 \times 3$ km^2, $40 \times 120 \times 20$ corner-point grid, several major faults

[1]Manzocchi et al., 2008

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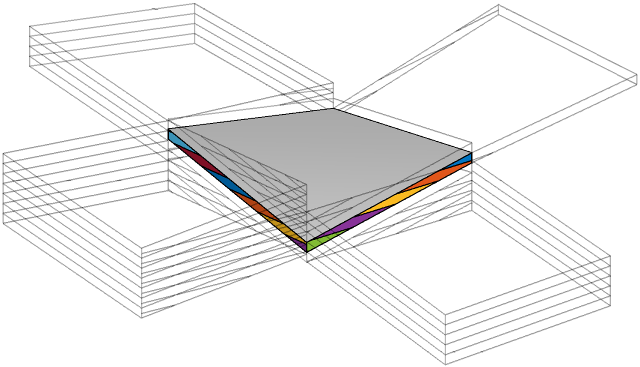
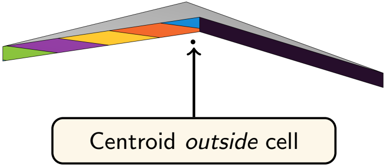
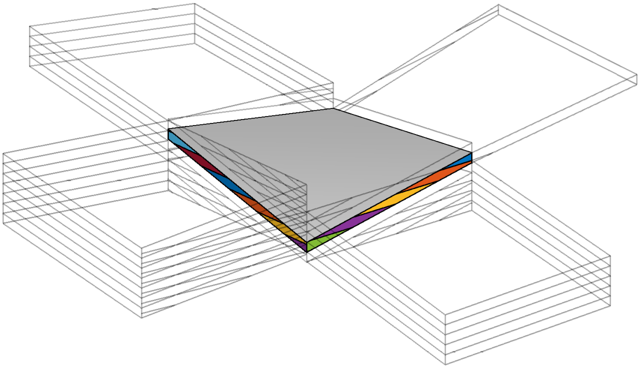
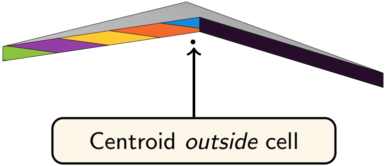




- Skewed and irregular cell geometries with complex topology – up to 20 neighbors
- Extreme aspect ratios – larges to smalles face area $= 2.6 \times 10^4$

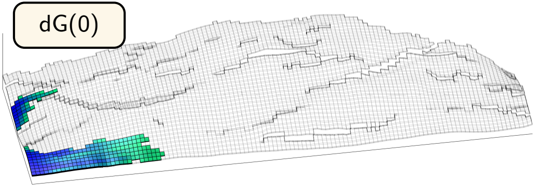
Subset of SAIGUP realization

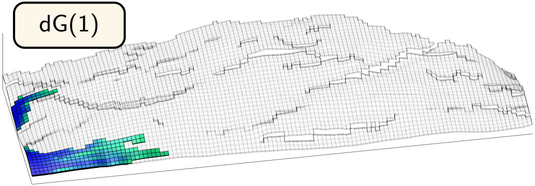


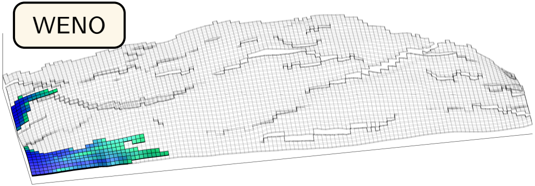


- WENO stencil robustness:
  - local affine transformation
  - use only one cell in each logical direction, chosen based on inteface area
- dG: bounding box basis functions greatly simplifies implementation

Example 2: Subset of SAIGUP realization







- Extract three top layers
- Pressure drop from west to east
- Equal viscosities, linear relperms
- Higher-order methods: visibly sharper and less diffusive profiles

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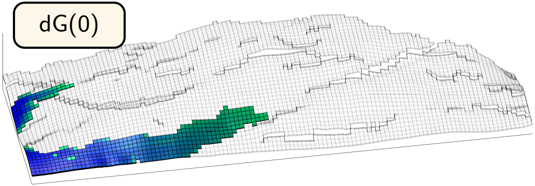

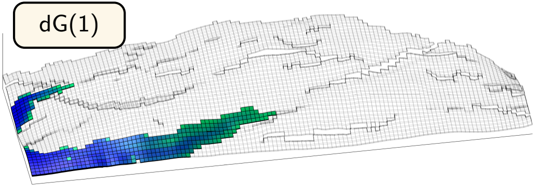

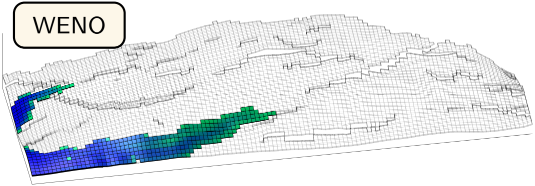

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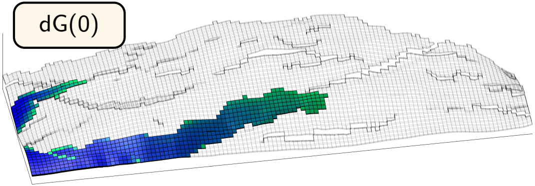

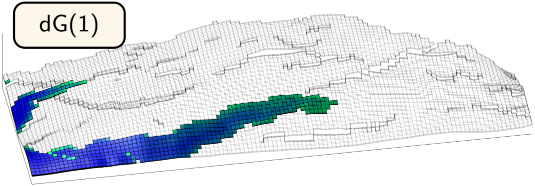

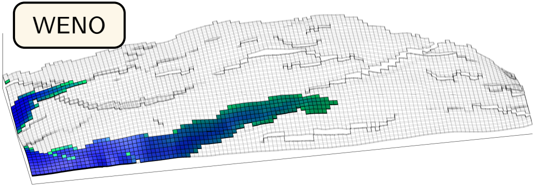

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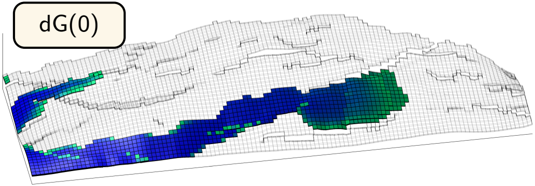

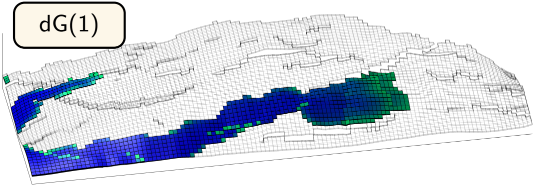

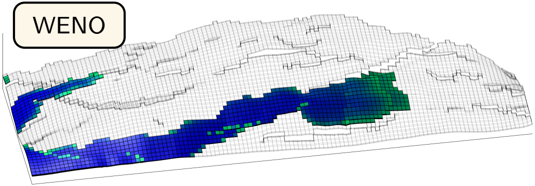

- Extract three top layers
- Pressure drop from west to east
- Equal viscosities, linear relperms
- Higher-order methods: visibly sharper and less diffusive profiles

Closing remarks

- Implementation complexity

WENO	Finite volume scheme $\rightarrow$ eas(y/ier) to build on existing simulator
dG	Requires heavy numerical machinery (basis functions, cubature rules, grid geometry ...)

- Computational complexity

WENO	Same number of unknowns (one per cell per phase/component), but *denser stencil*
dG	Large number of unknowns: $\binom{k+d}{k}$, observed slover nonlinear convergence

[1]Natvig and Lie, 2008 [2]Klemetsdal et al., 2018

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- **But:** dG stencil restriced to the cell and its upstream neighbors
  - Reorder grid cells based on intercell flux graph
    → transport subproblem can be solved *cell-by-cell* in topological order[1,2]

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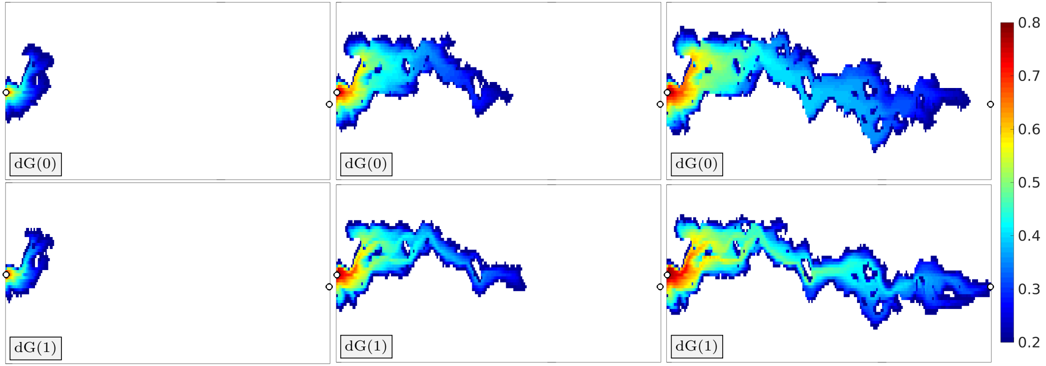
Nonlinear solver with optimal reordering



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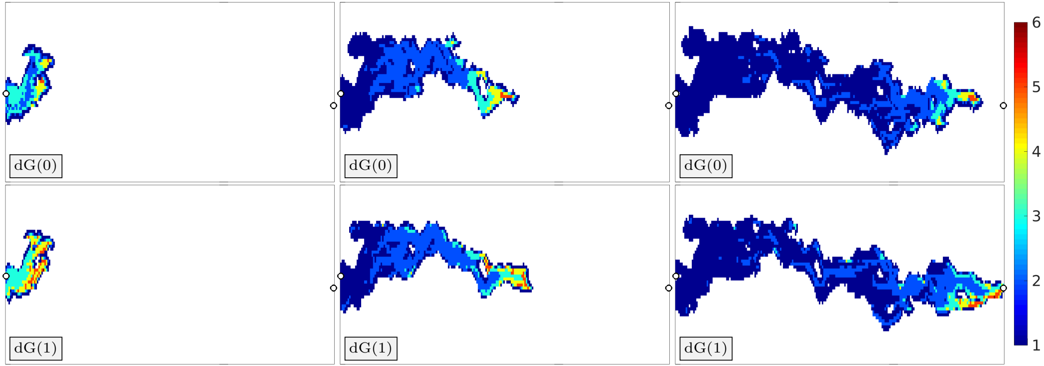

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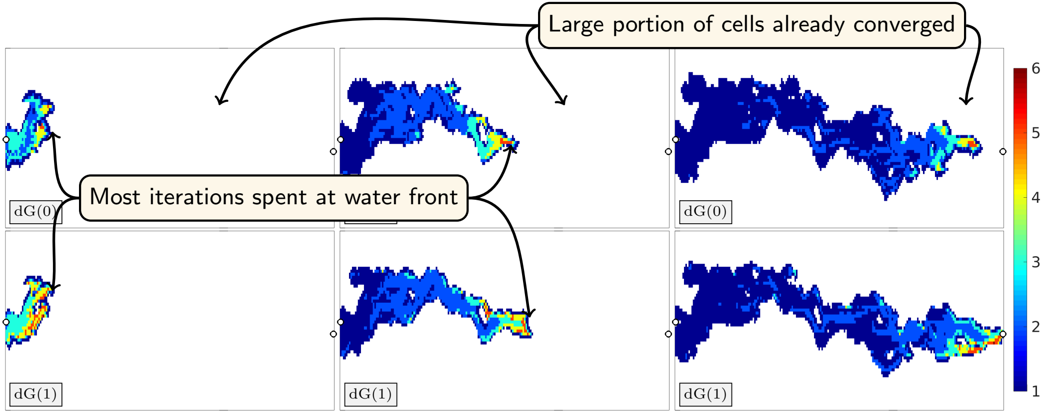

Acknowledgements

All simulations have been done using the
MATLAB Reservoir Simulation Toolbox (MRST)

mrst.no

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Why do we need implicit methods?

- Consider $\phi u_t + \vec{v} \cdot \nabla u = 0, \quad \vec{v} = -\frac{1}{\mu}\mathbf{K}\nabla p$
  - Large variations in petrophysical properties (porosity ϕ, permeability $\mathbf{K}$)
  - Large variations in $|\vec{v}|$: stagnant regions/high flow near wells

  $\rightarrow$ Explicit method: severe time-step restrictions, but significant smearing even with $CFL < 1$

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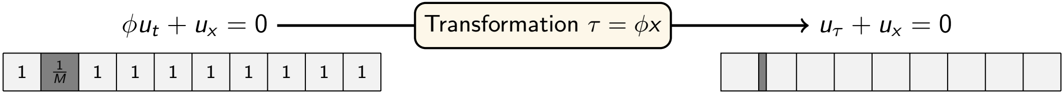

Modified equation (implicit/explicit first order) $q_t + q_\tau = \frac{1}{2}(\Delta\tau \pm \Delta t)q_{\tau\tau}$
$\rightarrow$ smearing of discontinuity across a width $\mathcal{O}\left(\sqrt{t(\Delta\tau \pm \Delta t)}\right)$
Scheme with CFL number ν (i.e., $\Delta t = \nu\frac{\Delta x}{M}$) gives overall smearing

$$\underbrace{\frac{9}{10}(\Delta x\phi \pm \Delta t)}_{\text{high-porosity region}} + \underbrace{\frac{1}{10M}\left(\frac{\Delta x\phi}{M} \pm \Delta t\right)}_{\text{low-porosity region}} = \frac{9\Delta x}{10}\left(1 \pm \frac{\nu}{M}\right) + \frac{\Delta x}{10M^2}(1 \pm \nu)$$

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