A Comparison of Consistent Discretizations for Elliptic Problems on Polyhedral Grids

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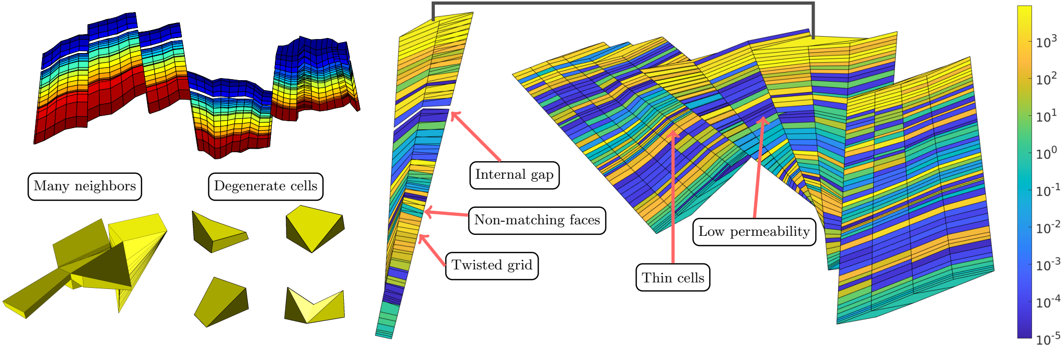
Introduction

- Subsurface reservoirs are complex: faults, fractures, complicated well paths, ...
- Simulation models often upscaled → polyhedral cells with full-tensor permeability

Industry-standard two-point flux approximation is generally *not consistent*



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In this work: Compare representative set of consistent methods within the same framework (MRST) with emphasis on robustness and computational efficiency

Consistent discretizations

- Incompressible, singe-phase, porous media flow

$$\nabla \cdot \underset{\text{Darcy velocity}}{\vec{v}} = \underset{\text{sources/sinks}}{q}, \qquad \text{where} \qquad \overbrace{\vec{v} = -\underset{\text{permeability}}{\mathbf{K}} \nabla \underset{\text{pressure}}{p}}^{\text{Darcy's law}}$$

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- Finite volume method: divide into cells Ω_i, integrate + divergence theorem

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Methods differ in how they approximate intercell fluxes v_{ij}

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Local conservation of mass

- Total mass flux across $\partial\Omega_i$ must equal net charge of fluids inside Ω_i ($v_{ij} = -v_{ji}$)

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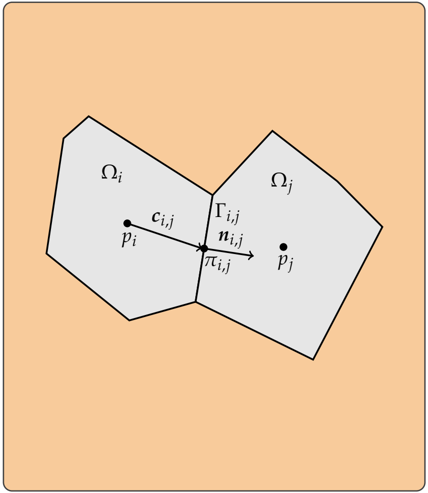
Computational efficiency

- Cost of assembling and solving the linear(ized) systems

Consistent discretizations

- TPFA: two-point expression $v_{ij} = T_{ij}(p_i - p_j)$
  - Monotone, only consistent for K-orthogonal grids



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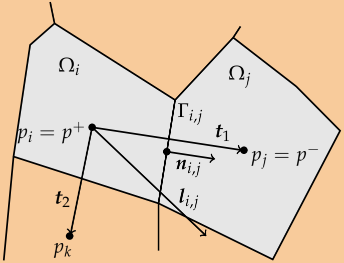

(Nikitin et al. [2014], Lipnikov et al. [2007], Le Potier [2009], Schneider et al. [2018], ...)

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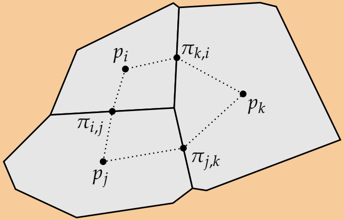

(Aavatsmark [2002], Edwards and Rogers [1994], Keilegavlen and Aavatsmark [2011], ...)

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- MFD: hybrid formulation, free stabilization parameter
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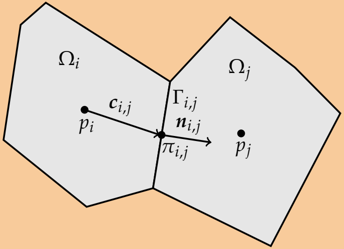

(Brezzi et al. [2005b], Lipnikov et al. [2009], Lie et al. [2012], da Veiga et al. [2014], ...)

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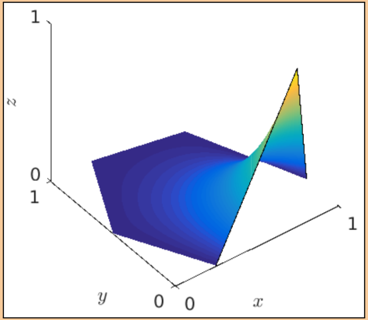
- VEM: FEM-type discretization for polytopal grids
  - Consistent, not monotone, here: non-conservative

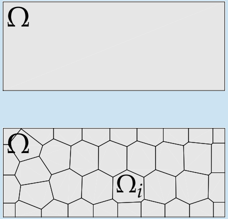


(Beirão da Veiga et al. [2013, 2014], Ahmad et al. [2013], Brezzi et al. [2014], ...)

Consistent discretizations

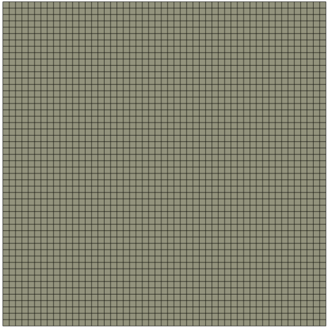
`dof`	Cell	Face	Node
TPFA	✓	✗	✗
NTPFA	✓	✗	✗
MPFA	✓	✗	✗
MFD	✓	✓	✗
VEM	✓(2nd)	✓(2nd)	✓

$$\begin{tikzcd} p & V \arrow[r, "\mathcal{A}"] \arrow[d, "\texttt{dof}"'] & V \arrow[d, "\texttt{dof}"] & q \\ \boldsymbol{p} & \mathbb{R}^n \arrow[r, "\mathcal{A}_h"'] & \mathbb{R}^n & \boldsymbol{q} \end{tikzcd}$$

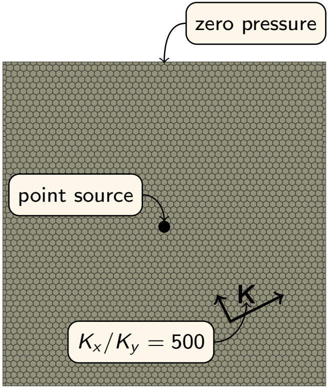


$\mathcal{A}_h$	Conservative	Consistent	Monotone	Linear	Higher-order
TPFA	✓	✗	✓	✓	✗
NTPFA	✓	✓	✓	✗	✗
MPFA	✓	✓	✗	✓	✗
MFD	✓	✓	✗	✓	✓
VEM	✗	✓	✗	✓	✓

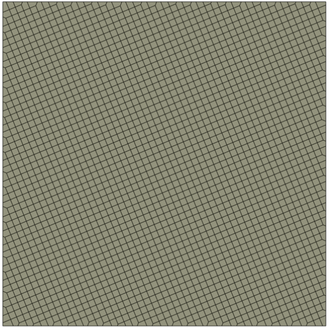
Example 1: Monotonicity



Cartesian grid



PEBI grid



Rotated Cartesian grid

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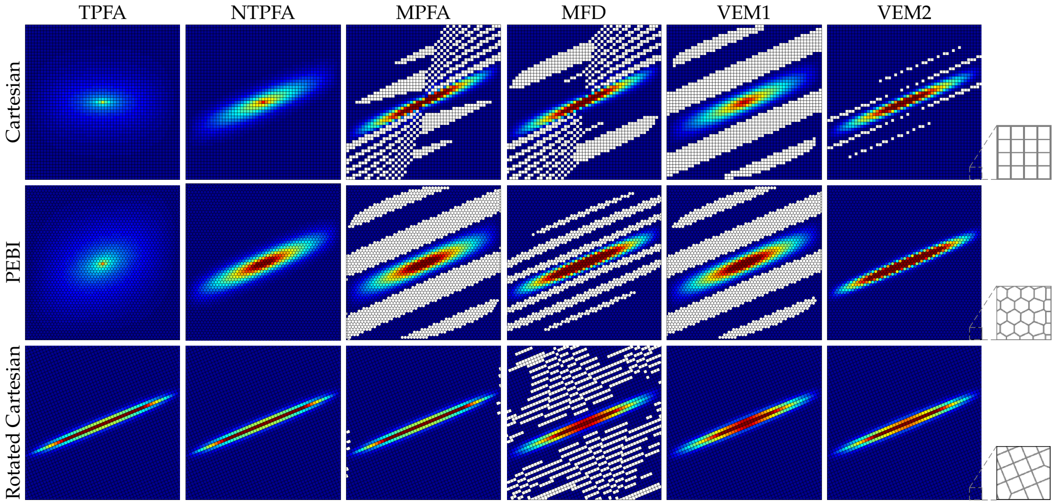

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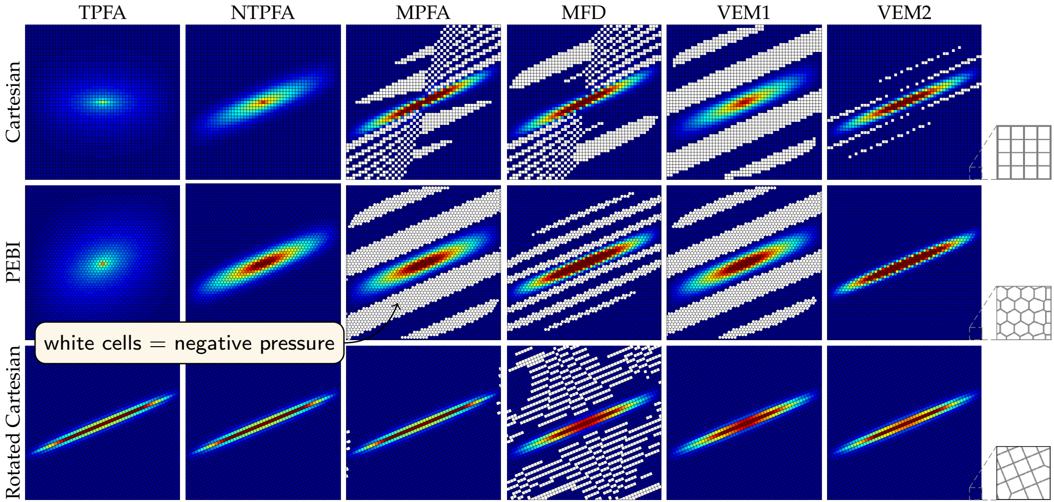

white cells = negative pressure

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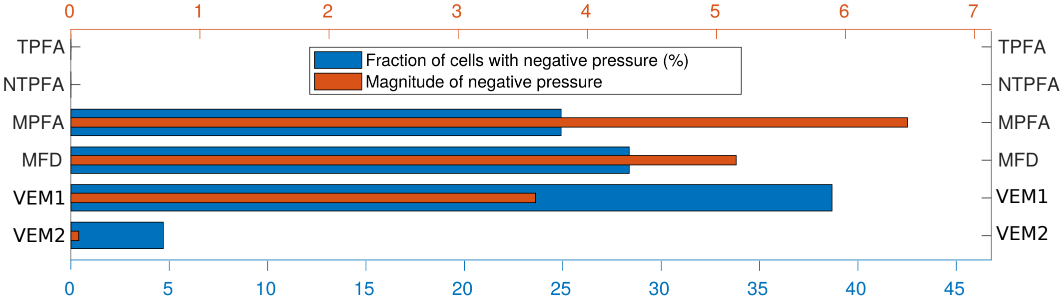


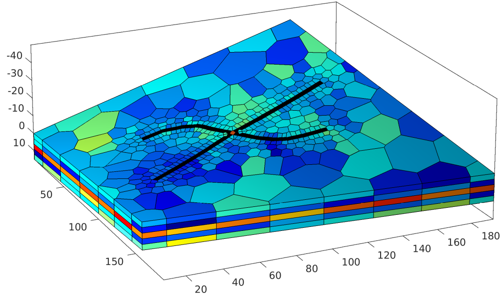
Higher-order VEM significantly better than the other linear methods,

Example 1: Monotonicity

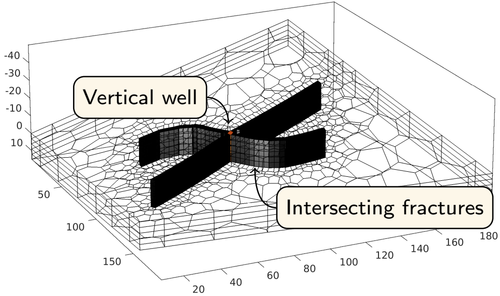
Method	dofs		nnz	ratio	cond
TPFA	cells	2601	12801	4.92	1.45e+03
NTPFA	cells	2601	17208	6.62	2.83e+03
MPFA	cells + outer faces	3009	23209	7.71	1.69e+03
MFD	faces	5100	35096	6.88	7.01e+03
VEM1	vertices	2704	22704	8.40	5.08e+04
VEM2	cells + faces + vertices	10609	162817	15.35	1.07e+06

Higher-order VEM significantly better than the other linear methods,
but also significantly more expensive

Example 2: Near-well simulation



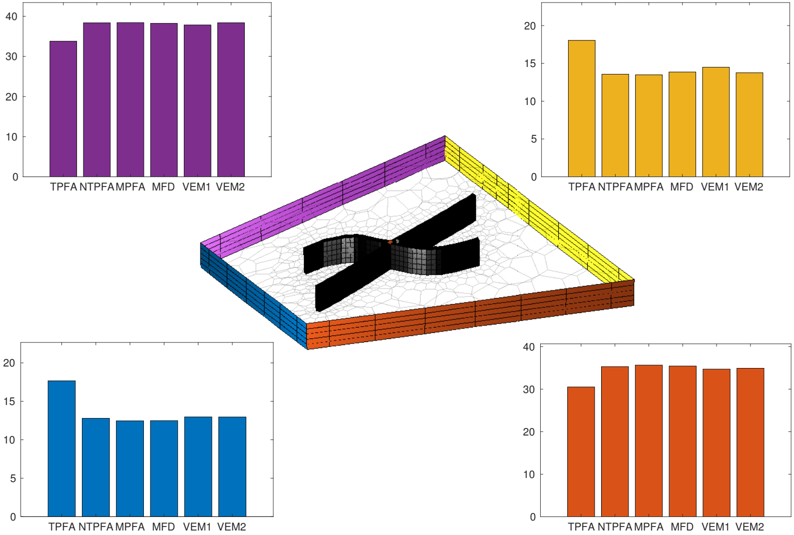
Permeability *x*-component



Setup

- Permeability anisotropy $K_x/K_y = 3$, rotated by $\pi/6$ in xy-plane
- Well injects 1 PV over 0.1 years, no-flow on top/bottom, fixed pressure on vertical sides

Example 2: Near-well simulation



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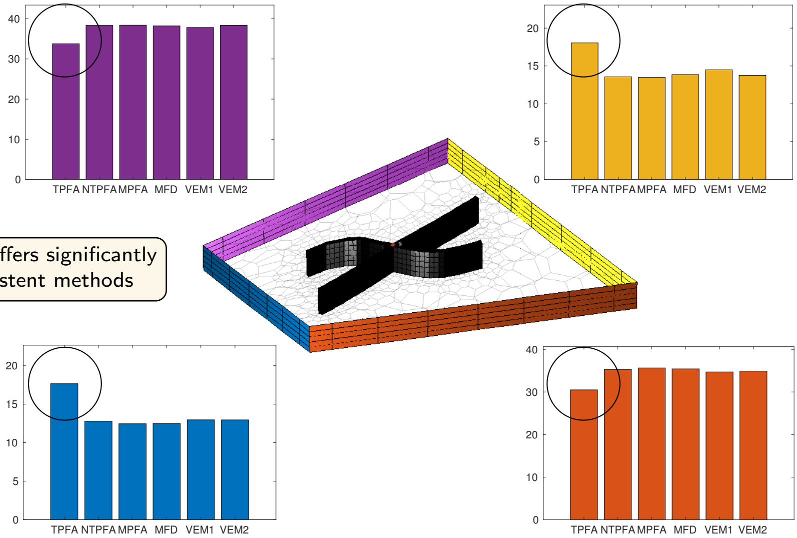
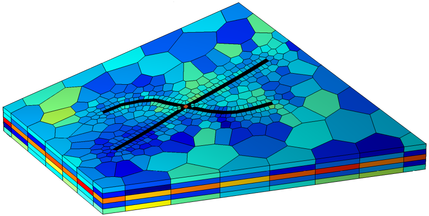


TPFA flux differs significantly
from consistent methods

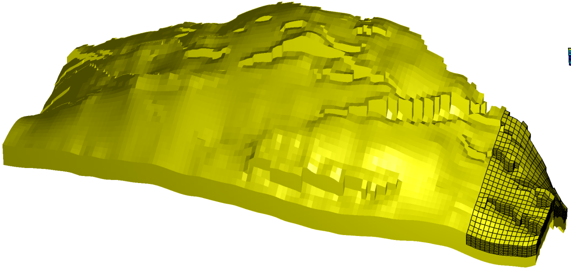
Example 2: Near-well simulation

Method	dof	nnz	ratio	cond
TPFA	2 465	19 809	8.04	1.11e+04
NTPFA	2 465	33 608	13.63	3.26e+05
MPFA	8 507	98 579	11.59	6.74e+04
MFD	9 658	130 438	13.51	2.94e+09
VEM1	5 274	170 618	32.35	2.22e+11
VEM2	30 173	2 495 409	82.70	1.44e+12

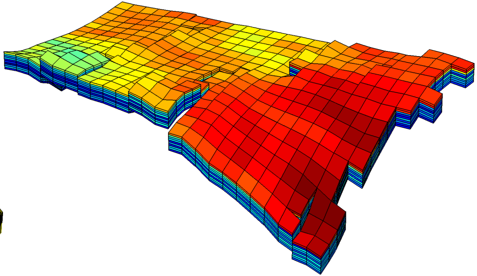


Differences observed in 2D are even more severe in 3D

Example 3: Multiphase flow



Full model



Subset of model

SAIGUP study: Geomodel of shallow-marine oil reservoir with several major faults and mud-rapes, posed on cornerpoint grid (Manzocchi et al. [2008])

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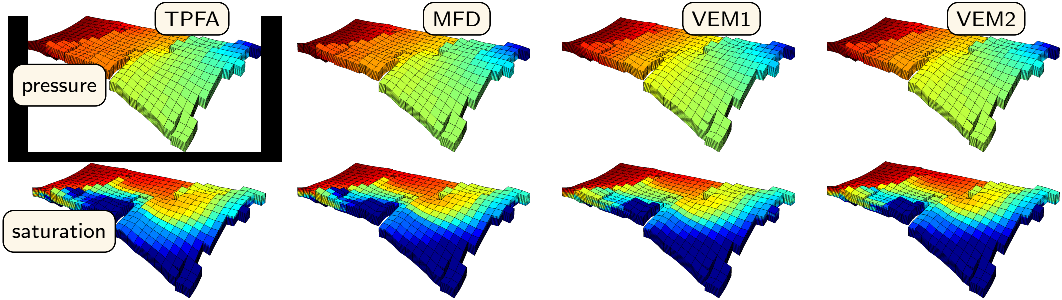

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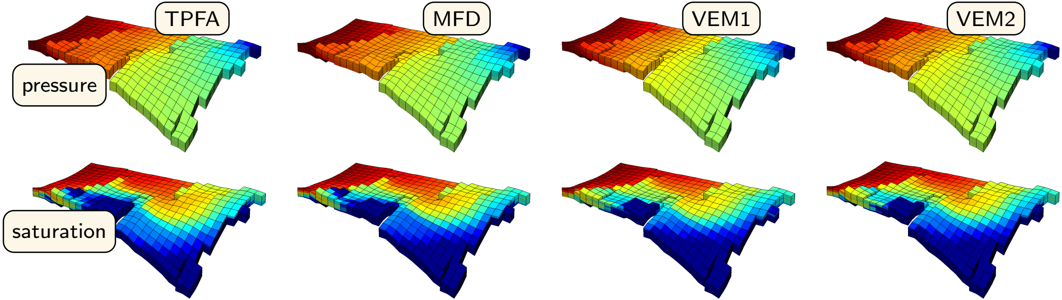

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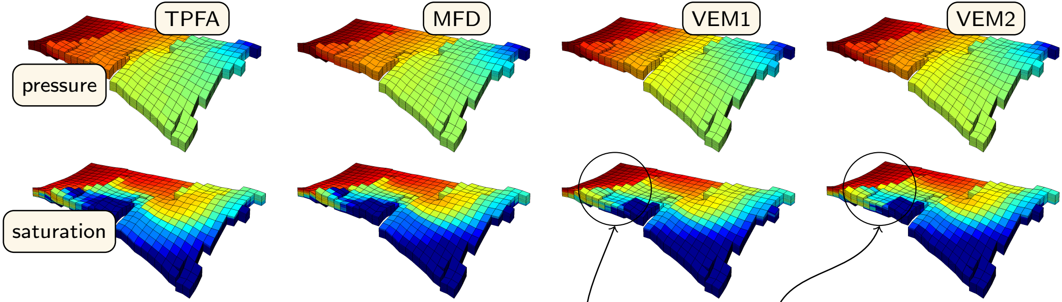
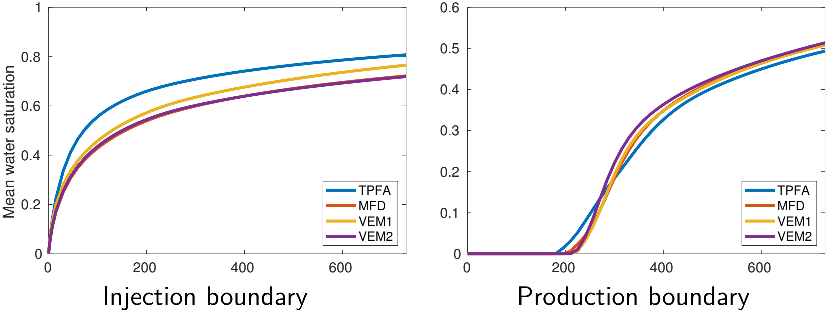


VEM solutions differ significantly due to postprocessing of fluxes

Example 3: Multiphase flow

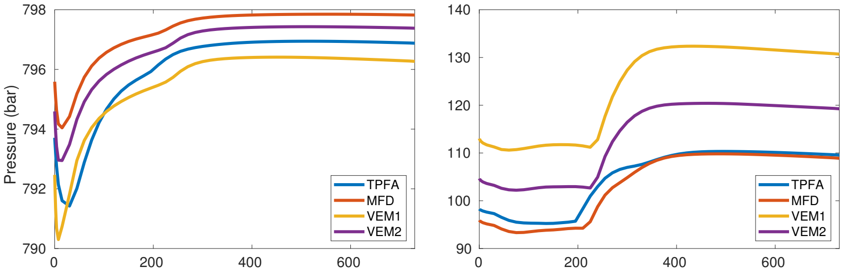
Mean water cut vs. time



- TPFA: larger saturation along injection boundary, and earlier breakthrough

Example 3: Multiphase flow

Mean pressure vs. time



Injection boundary

Production boundary

- TPFA: larger saturation along injection boundary, and earlier breakthrough
- Up to 15% difference between pressures, also for consistent methods
  - VEM postprocessing before transport may introduce artifacts in flow field

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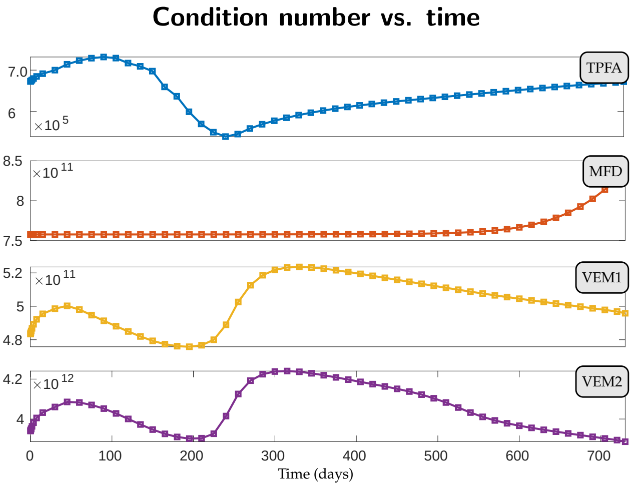
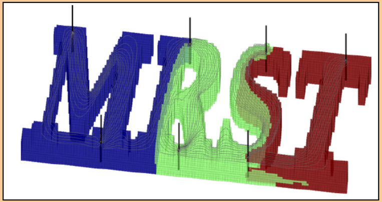


Conclusions

- TPFA inconsistent, grid effects, but monotone and matrices with low condition numbers
- Consistent methods: convergent, less grid effects, but monotonicity issues and denser, more ill-conditioned matrices
- MFD easy to implement, flexible wrt. grids, but not cell centered
- MPFA more difficult to implement and challenged by co-planar surface patches
- NTPFA promising, but not yet sufficiently robust



`mrst.no`

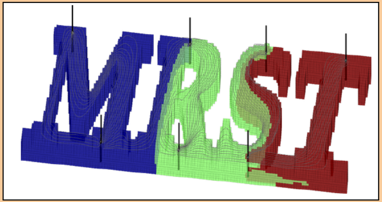
Paper source code and more examples:

`git@bitbucket.org:strene/compare-elliptic.git`

Conclusions

General advice

- Use *multiple* consistent methods to assess error from anisotropic permeability and grid orientation



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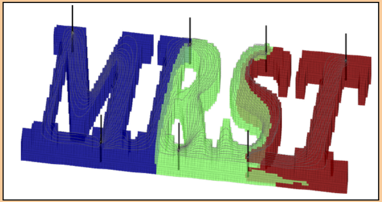
Further work

- Multiphase: use flow diagnostics tools
  - sweep, drainage regions, well pairs, TOF, etc.
  - #/size of connected components in flux graph
- Effect on linear and nonlinear solver performance
- How does discretization affect transport solver?



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Acknowledgements

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Extra