Additive Schwarz Preconditioned Exact Newton Method as a Nonlinear Preconditioner for Multiphase Porous Media Flow

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Introduction: Domain decomposition methods in reservoir simulation

	Variable	Space
Linear		
Nonlinear		

Introduction: Domain decomposition methods in reservoir simulation

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Linear	✓	✓
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- Linear domain decomposition
  - Well-established means to construct scalable algorithms
    Wallis et al. [1985], Killough and Wheeler [1987], Stueben [2001], Lie et al. [2017]

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- Nonlinear domain decomposition
  - Popular in variable domain $\rightarrow$ sequential splitting
    Watts [1986], Trangenstein and Bell [1989], Jenny et al. [2006], Møyner and Lie [2016]

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    Watts [1986], Trangenstein and Bell [1989], Jenny et al. [2006], Møyner and Lie [2016]
  - Not so much in spatial domain
  - However: problem equations are strongly coupled, highly nonlinear, unbalanced
    $\rightarrow$ Spatial domain decomposition is an excellent nonlinear a preconditioner

Introduction: Domain decomposition methods in reservoir simulation

Additive Schwarz Preconditioned Inexact Newton (ASPIN) (Cai and Keyes [2002])

- Spatial domain decomposition (Liu et al. [2013], Skogestad et al. [2013] (two-phase flow))
- Variable domain decomposition (Li et al. [2019] (MS wells), Wong [2018] (geothermal))
- Related: Dolean et al. [2016] (Forchheimer equation), Liu et al. [2013] (Multiplicative SPIN)

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In this work: apply nonlinear domain decomposition preconditioning (NLDDP) (exact/inexact) to complex problems with realistic geology and fluid physics

Nonlinear domain decomposition preconditioning

- Conservation of mass of a component i on discrete, implicit form

$$\mathbf{R}_i^{n+1} = \frac{1}{\Delta t^n}(\underbrace{\mathbf{M}_i^{n+1}}_{\text{Mass}} - \mathbf{M}_i^n) + \mathtt{div}(\underbrace{\mathbf{V}_i^{n+1}}_{\text{Flux}}) - \underbrace{\mathbf{Q}_i^{n+1}}_{\text{Sources/sinks}} = 0$$

- `div`: discrete divergence operator (two-point, multipoint, mimetic, etc.)
  - In this work: linear two-point flux approximation

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- `div`: discrete divergence operator (two-point, multipoint, mimetic, etc.)
  - In this work: linear two-point flux approximation
- Gather in nonlinear system $\mathbf{R}(\mathbf{u}) = \mathbf{0}$, linearize, and neglect higher-order terms $\rightarrow$ iterative Newton's method

$$\mathbf{u}^{k+1} = \mathbf{u}^k + \Delta\mathbf{u}, \quad -\frac{\partial \mathbf{R}}{\partial \mathbf{u}}\Delta\mathbf{u} = \mathbf{R}(\mathbf{u}^k)$$

Nonlinear domain decomposition preconditioning

Nonlinear preconditioning with two subdomains

- Partition unknowns $\mathbf{u}$ into two non-overlapping subdomains

$$\mathbf{R}(\mathbf{u}) = (\mathbf{R}_1(\mathbf{u}_1, \mathbf{u}_2), \mathbf{R}_2(\mathbf{u}_1, \mathbf{u}_2)) = \mathbf{0},$$

- Define solution operator $\mathcal{L}(\mathbf{u}) = (\mathcal{L}_1(\mathbf{u}), \mathcal{L}_2(\mathbf{u}))$, where

$$\mathbf{R}_1\big(\mathcal{L}_1(\mathbf{u}), \mathbf{u}_2\big) = \mathbf{0}, \quad \text{and} \quad \mathbf{R}_2\big(\mathbf{u}_1, \mathcal{L}_2(\mathbf{u})\big) = \mathbf{0}.$$

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Equivalent, fixed-point formulation of $\mathbf{R}(\mathbf{u}) = \mathbf{0}$

Find $\mathbf{u}$ so that $\mathbf{u} = \mathcal{L}(\mathbf{u})$, or $\mathbf{F}(\mathbf{u}) \equiv \mathbf{u} - \mathcal{L}(\mathbf{u}) = \mathbf{0}$

Nonlinear domain decomposition preconditioning

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  - Acceleration: Aitken, Anderson, quasi-Newton (Jiang and Tchelepi [2019])

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- Fixed-point schemes tend to have poor convergence properties
  - Acceleration: Aitken, Anderson, quasi-Newton (Jiang and Tchelepi [2019])
- Here: apply Newton's method directly to $\mathbf{F}(\mathbf{u})$:

$$\mathbf{u}^{k+1} = \mathbf{u}^k + \Delta\mathbf{u}, \quad -\frac{\partial \mathbf{F}}{\partial \mathbf{u}}\Delta\mathbf{u} = \mathbf{F}(\mathbf{u}^k), \quad \text{where} \quad \frac{\partial \mathbf{F}}{\partial \mathbf{u}} = \mathbf{I} - \begin{bmatrix} \frac{\partial \mathcal{L}_1}{\partial \mathbf{u}} \\ \frac{\partial \mathcal{L}_2}{\partial \mathbf{u}} \end{bmatrix}$$

  - Challenge: $\mathbf{F}$ implicitly defined through operator $\mathcal{L}$ – how to compute $\partial\mathbf{F}/\partial\mathbf{u}$?

Nonlinear domain decomposition preconditioning

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- Use that $\mathbf{R}_1(\mathcal{L}_1(\mathbf{u}), \mathbf{u}_2) = \mathbf{0}$ to find

$$\frac{\partial \mathbf{R}_1}{\partial \mathbf{u}} = \frac{\partial \mathbf{R}_1}{\partial \mathbf{u}_1} \frac{\partial \mathcal{L}_1}{\partial \mathbf{u}} + \frac{\partial \mathbf{R}_1}{\partial \mathbf{u}_2} \frac{\partial \mathbf{u}_2}{\partial \mathbf{u}} = \mathbf{0}$$

- Rearrange to obtain

$$\frac{\partial \mathcal{L}_1}{\partial \mathbf{u}} = -\left(\frac{\partial \mathbf{R}_1}{\partial \mathbf{u}_1}\right)^{-1} \frac{\partial \mathbf{R}_1}{\partial \mathbf{u}_2} \frac{\partial \mathbf{u}_2}{\partial \mathbf{u}} \quad \text{and} \quad \frac{\partial \mathcal{L}_2}{\partial \mathbf{u}} = -\left(\frac{\partial \mathbf{R}_2}{\partial \mathbf{u}_2}\right)^{-1} \frac{\partial \mathbf{R}_2}{\partial \mathbf{u}_1} \frac{\partial \mathbf{u}_1}{\partial \mathbf{u}}$$

- Natural extension to m subdomains

Nonlinear domain decomposition preconditioning

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Find $\mathbf{u}$ so that $\mathbf{u} = \mathcal{L}(\mathbf{u})$, or $\mathbf{F}(\mathbf{u}) \equiv \mathbf{u} - \mathcal{L}(\mathbf{u}) = \mathbf{0}$

- Jacobian $\partial\mathbf{F}/\partial\mathbf{u}$ generally dense $\rightarrow$ expensive to build, challenging to precondition
- Breakdown of Jacobian blocks reveals that

$$\frac{\partial \mathbf{F}}{\partial \mathbf{u}} = \begin{bmatrix} \frac{\partial \mathbf{R}_1}{\partial \mathbf{u}_1} & \mathbf{0} \\ \mathbf{0} & \frac{\partial \mathbf{R}_2}{\partial \mathbf{u}_2} \end{bmatrix}^{-1} \frac{\partial \mathbf{R}}{\partial \mathbf{u}} \equiv \mathbf{D}^{-1} \frac{\partial \mathbf{R}}{\partial \mathbf{u}}$$

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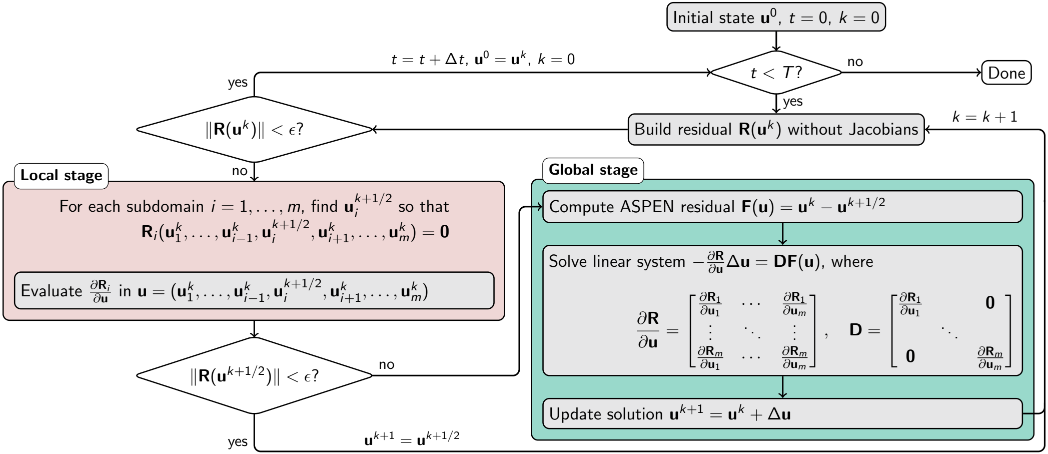
Original problem Jacobian (almost)

$\rightarrow$ Can interpret linearized system as

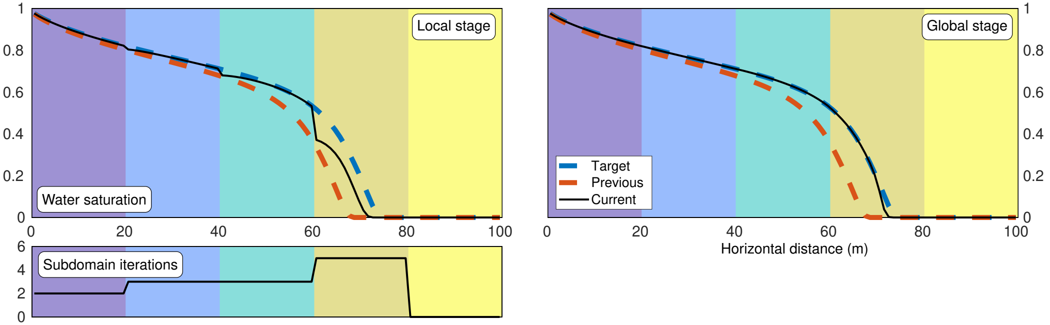
$$-\frac{\partial \mathbf{F}}{\partial \mathbf{u}} \Delta \mathbf{u} = \mathbf{F}(\mathbf{u}) \quad \Longleftrightarrow \quad -\frac{\partial \mathbf{R}}{\partial \mathbf{u}} \Delta \mathbf{u} = \mathbf{D}\mathbf{F}(\mathbf{u}).$$

- We're back on home ground – we know what preconditioners to use!

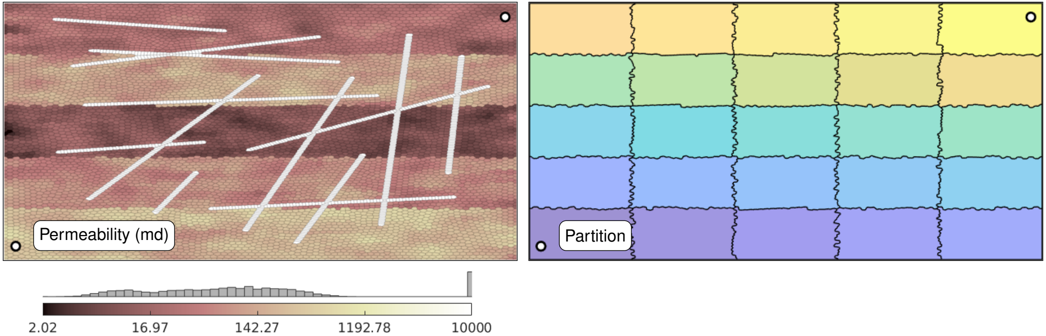


Initial state $\mathbf{u}^0$, $t = 0$, $k = 0$
$t < T$?
no
Done
yes
$t = t + \Delta t$, $\mathbf{u}^0 = \mathbf{u}^k$, $k = 0$
yes
Build residual $\mathbf{R}(\mathbf{u}^k)$ without Jacobians
$k = k + 1$
$\|\mathbf{R}(\mathbf{u}^k)\| < \epsilon$?
no
Local stage
For each subdomain $i = 1, \ldots, m$, find $\mathbf{u}_i^{k+1/2}$ so that $\mathbf{R}_i(\mathbf{u}_1^k, \ldots, \mathbf{u}_{i-1}^k, \mathbf{u}_i^{k+1/2}, \mathbf{u}_{i+1}^k, \ldots, \mathbf{u}_m^k) = \mathbf{0}$
Evaluate $\frac{\partial \mathbf{R}_i}{\partial \mathbf{u}}$ in $\mathbf{u} = (\mathbf{u}_1^k, \ldots, \mathbf{u}_{i-1}^k, \mathbf{u}_i^{k+1/2}, \mathbf{u}_{i+1}^k, \ldots, \mathbf{u}_m^k)$
$\|\mathbf{R}(\mathbf{u}^{k+1/2})\| < \epsilon$?
no
yes
$\mathbf{u}^{k+1} = \mathbf{u}^{k+1/2}$
Global stage
Compute ASPEN residual $\mathbf{F}(\mathbf{u}) = \mathbf{u}^k - \mathbf{u}^{k+1/2}$
Solve linear system $-\frac{\partial \mathbf{R}}{\partial \mathbf{u}} \Delta \mathbf{u} = \mathbf{D}\mathbf{F}(\mathbf{u})$, where
$\frac{\partial \mathbf{R}}{\partial \mathbf{u}} = \begin{bmatrix} \frac{\partial \mathbf{R}_1}{\partial \mathbf{u}_1} & \cdots & \frac{\partial \mathbf{R}_1}{\partial \mathbf{u}_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial \mathbf{R}_m}{\partial \mathbf{u}_1} & \cdots & \frac{\partial \mathbf{R}_m}{\partial \mathbf{u}_m} \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} \frac{\partial \mathbf{R}_1}{\partial \mathbf{u}_1} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \frac{\partial \mathbf{R}_m}{\partial \mathbf{u}_m} \end{bmatrix}$
Update solution $\mathbf{u}^{k+1} = \mathbf{u}^k + \Delta \mathbf{u}$

Example 1: Buckley-Leverett displacement



Example 2: Fractured reservoir



- 1000×500 m^2 reservoir with thirteen fractures on layered, low-perm background[1]
- Two-phase liquid-gas model with n-decane, carbon dioxide, and methane
- Simulate injection of n-decane and carbon dioxide mixture over 2555 days

[1] Slightly modified from Møyner and Tchelepi [2018]

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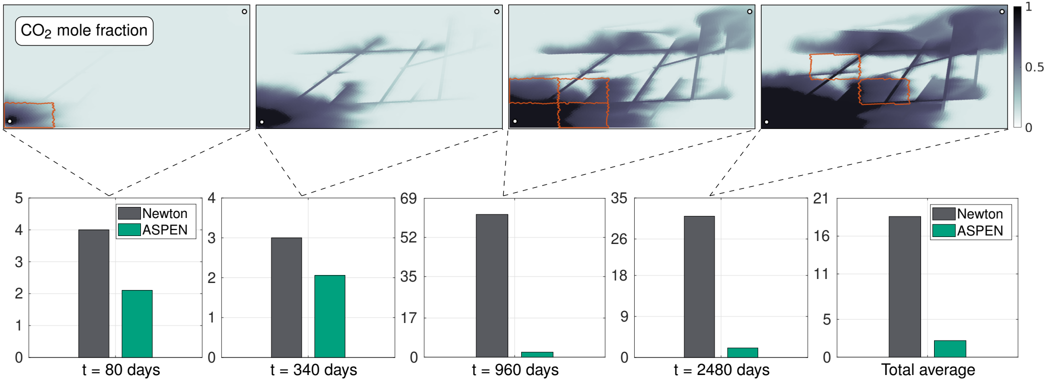

- Subdomains with more than seven iterations in total outlined in red

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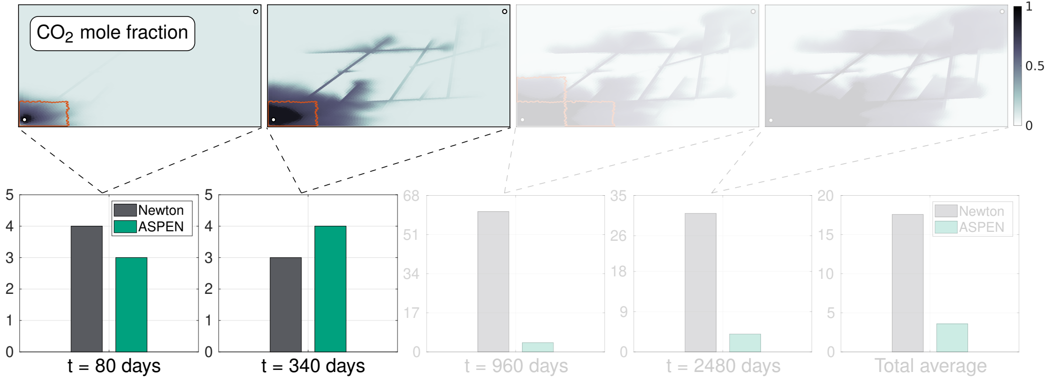

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- Steady convergence for both solvers during the first 900 days of injection

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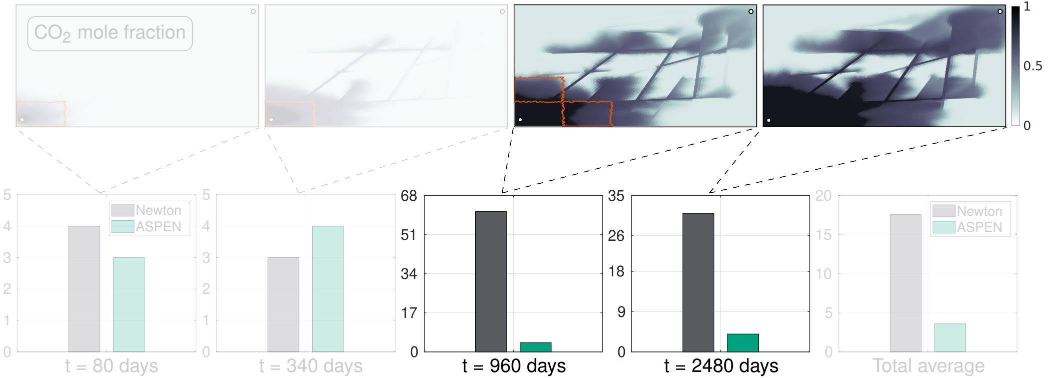

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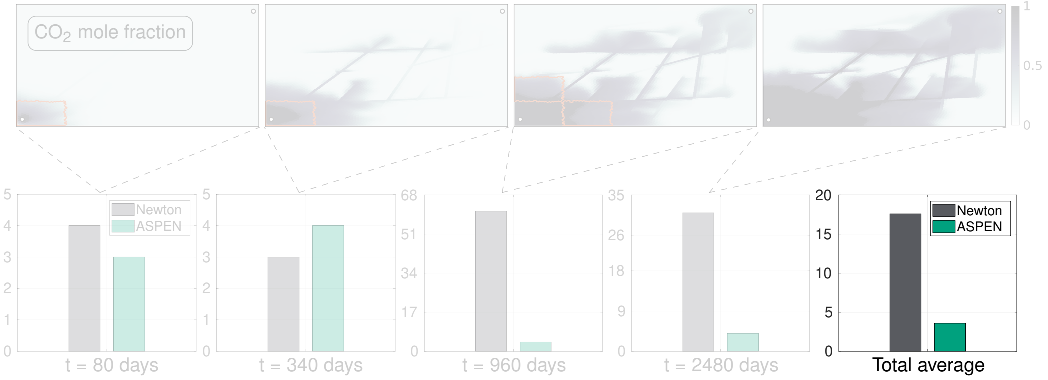

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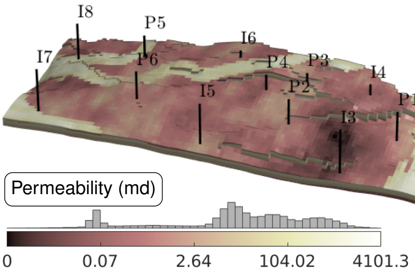
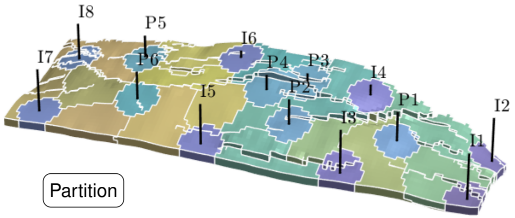
Example 4: Field-scale model (SAIGUP)





- Shallow-marine oil reservoir, modeled in the SAIGUP study (Manzocchi et al. [2008])
- Spans lateral area of $\sim 9 \times 3$ km^2, $40 \times 120 \times 20$ corner-point grid, several major faults
- Simulate 30 years of water injection with slightly compressible two-phase oil-water model
- Subdomains constructed by combining METIS partition with tubes around each well

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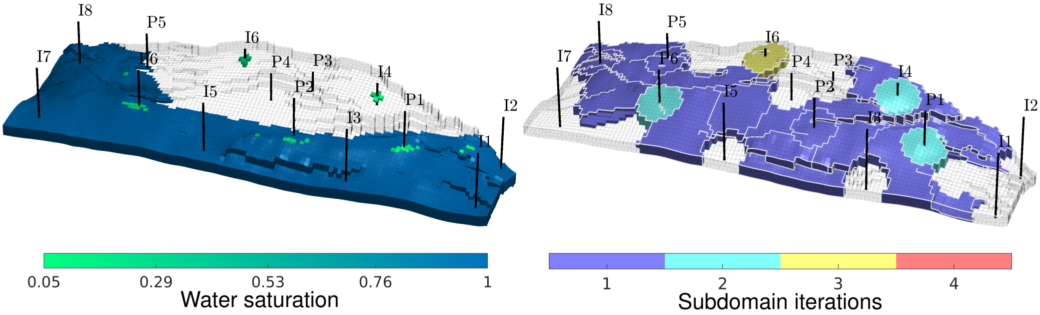

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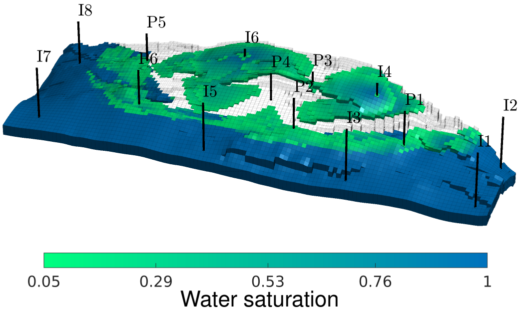

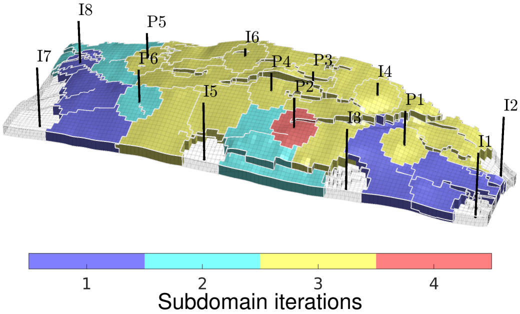

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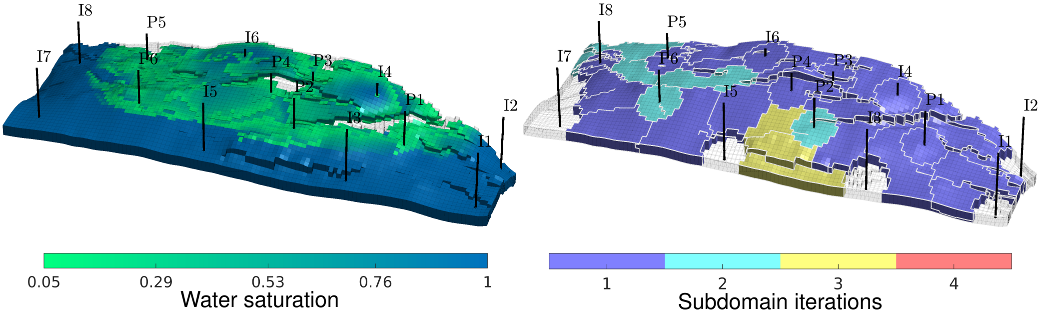

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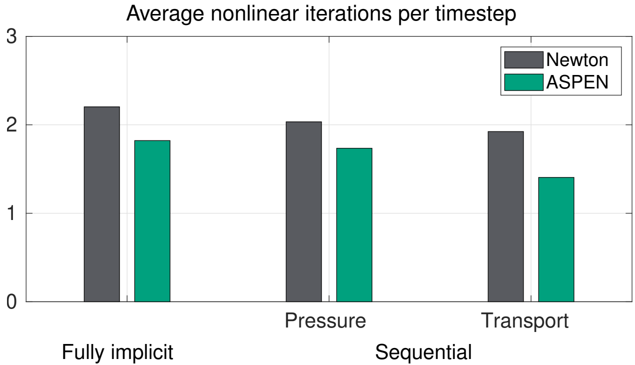
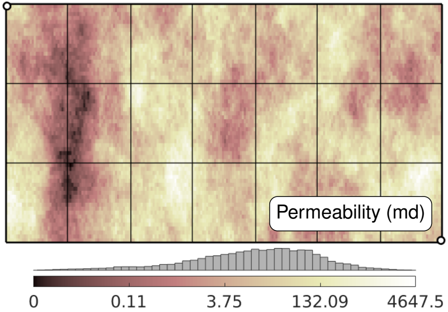


- Moderate CFL, simple fluid physics $\rightarrow$ near-optimal global nonlinear solver performance
- ASPEN nevertheless uses less iterations than global solver
  - Arguably equally important as excellent performance on challenging corner cases

Example 5: Water-alternating gas injection (WAG)



- First layer of SPE10 Model 2, three-phase, six-component fluid (C1, 3, 6, 10, 15, and 20)
- WAG injection: C1 gas for 5000 days + water for 5000 days + C1 gas for 5000 days[1]
- Simulate four setups with target timesteps of 25, 50, 100 and 200 days

[1] Example from Moncorgé et al. [2018]

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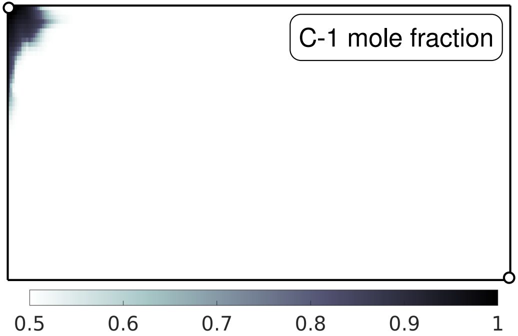

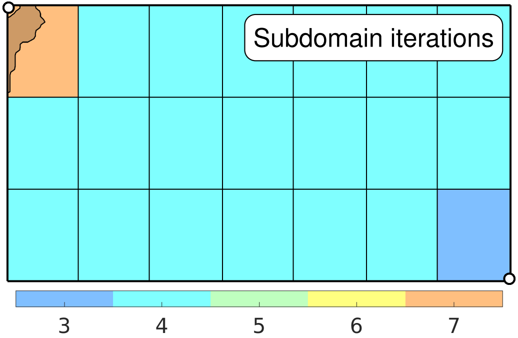

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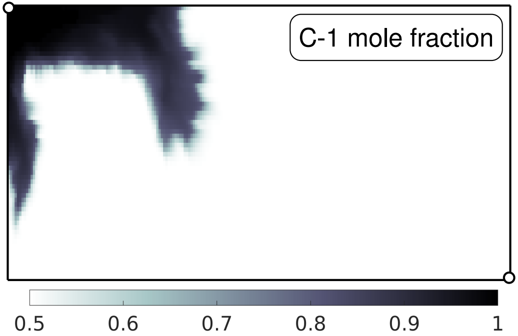

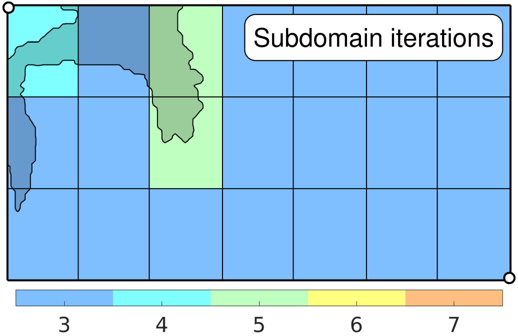

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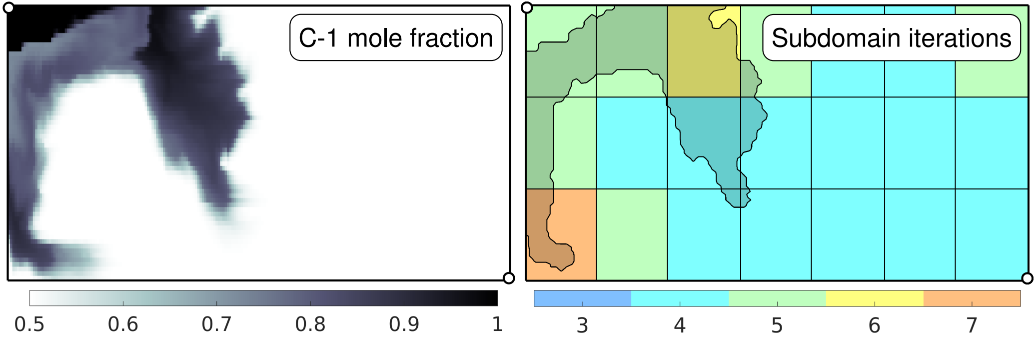

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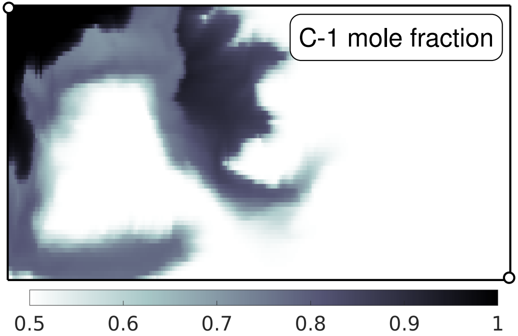

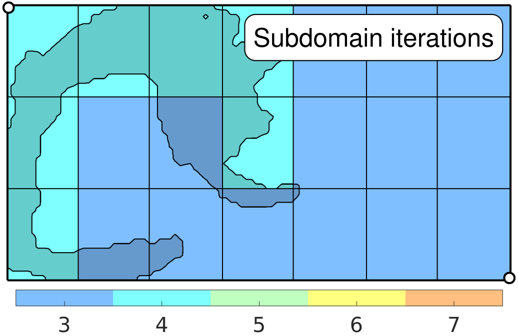

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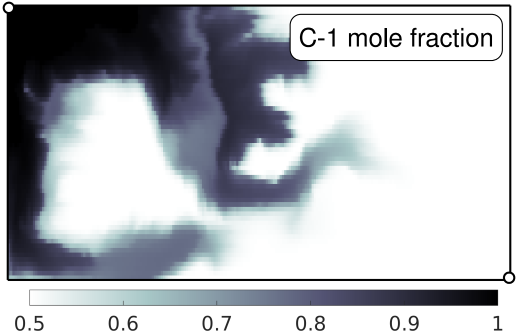

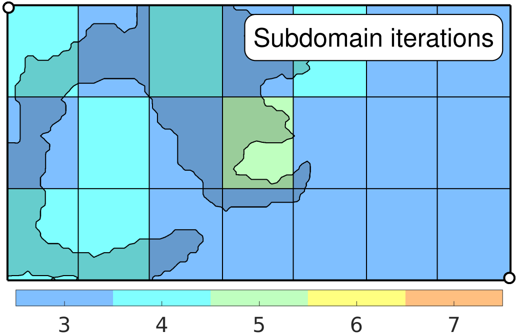

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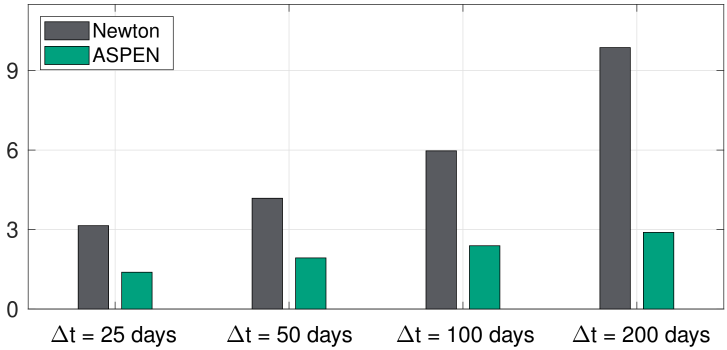

- Significant increase in global solver iterations with Δt – only slight increase for ASPEN
- Robustness with respect to Δt due to local control of nonlinear solution process

Concluding remarks

Conclusions

- Nonlinear domain decomposition preconditioner applicable to realistic problems
- Significant reduction in nonlinear iterations for wide range of examples
- Robust with respect to timestep size
- Reformulation enables using established iterative linear solvers

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Future work

- Multiplicative NLDDP for transport – reordering (Natvig and Lie [2008])
- Further utilize localization
  - Local timestepping techniques (Linga et al. [2020])
  - Adaptive sequential fully-implicit methods (Møyner and Moncorgé [2019])
  - Dynamic coarsening (Klemetsdal and Lie [2020])

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Extra: Nonlinear domain decomposition preconditioning

Equivalent, fixed-point formulation of $\mathbf{R}(\mathbf{u}) = \mathbf{0}$

Find $\mathbf{u}$ so that $\mathbf{u} = \mathcal{L}(\mathbf{u})$, or $\mathbf{F}(\mathbf{u}) \equiv \mathbf{u} - \mathcal{L}(\mathbf{u}) = \mathbf{0}$

Natural extension to m subdomains $\mathbf{u} = (\mathbf{u}_1, \ldots, \mathbf{u}_m)$, $\big(\mathbf{R}_1(\mathbf{u}), \ldots, \mathbf{R}_m(\mathbf{u})\big)$

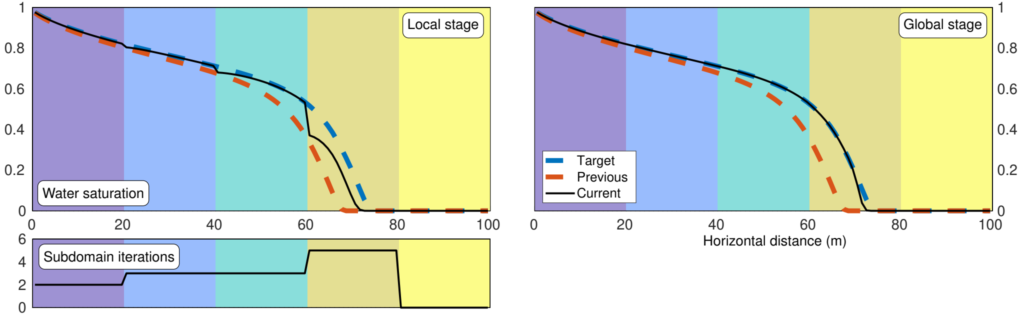
- Corresponding solution operators $\mathcal{L}_1, \ldots, \mathcal{L}_m$

$$\mathbf{R}_i\big(\mathbf{u}_1, \ldots, \mathbf{u}_{i-1}, \mathcal{L}_i(\mathbf{u}), \mathbf{u}_{i+1}, \ldots, \mathbf{u}_m\big) = \mathbf{0}$$

- As for the two-subdomain-case, we get

$$\frac{\partial \mathcal{L}_i}{\partial \mathbf{u}} = -\left(\frac{\partial \mathbf{R}_i}{\partial \mathbf{u}_i}\right)^{-1} \left(\sum_{j=1, j\neq i}^{m} \frac{\partial \mathbf{R}_i}{\partial \mathbf{u}_j} \frac{\partial \mathbf{u}_j}{\partial \mathbf{u}}\right).$$

Extra: Buckley-Leverett displacement (Example 1)

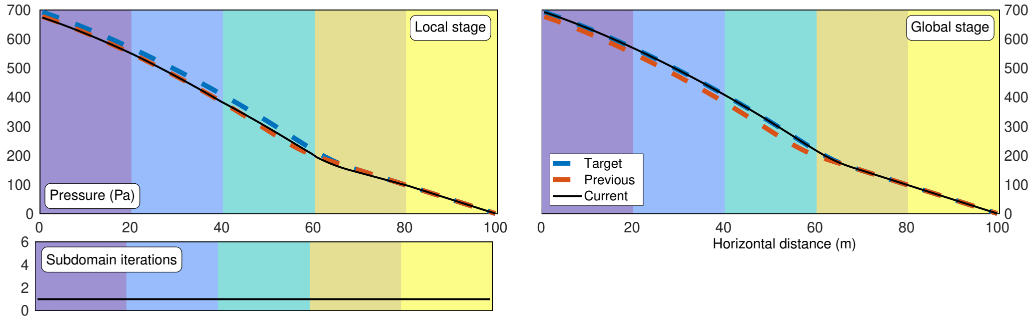


Horizontal 1D channel, CFL $= 1$, quadratic relative permeabilities, equal viscosities

Transport subproblem

- Visual kinks at subdomain boundaries after local solve
- Almost converged after additional global solve

Extra: Buckley-Leverett displacement (Example 1)



Horizontal 1D channel, CFL $= 1$, quadratic relative permeabilities, equal viscosities

Pressure subproblem

- Local solve far from target solution
- Global solve effectively resolves long-range interactions

Extra: Buckley-Leverett displacement (Example 1)

Nonlinear iterations per timestep